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A study on geometrical diagrams for 3-D objects in pre-modern East-Asian mathematical texts

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Abstract

In this paper, we discuss features of geometrical diagrams in different cultures, especially representations of 3-D objects in East Asia. There is no Chinese diagram in the first millennium or earlier that survives to this day. In the second millennium, there are some interesting features. In the first half of the second millennium, Chinese diagrams for 3-D objects are usually contextualised; in the second half, the diagrams became more abstract. Korean and Japanese mathematicians used very creative ways, such as parallel line segments or shadows, to draw 3-D objects. Japanese scholars also put a middle curve in a 2-D diagram to create the sense of the third dimension. For *sangaku* diagrams, they also put different colours on the two regions divided by the middle curve, as if those regions are different "faces".

§1. Introduction

Diagrams in pre-modern mathematical texts are important artefacts for historians to understand how ancient cultures represent physical objects or abstract shapes on flat faces such as cave walls or pieces of paper. For physical objects, pre-modern makers of

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the diagrams had to decide which characteristics of an object were to be emphasised and put on the diagram, and how those characteristics should be represented. These diagrams could never be visually identical to those physical objects since drawings were not photography. For abstract shapes, makers of the diagrams also had to decide how those shapes, or some combinations of abstract geometrical concepts, could be visually presented. Again, these diagrams might or might not show all necessary characteristics of a geometrical entity in, for instance, the Euclidean plane. A line in Euclidean geometry has only length but no width or depth, but obviously no pencil could be used to draw that kind of lines.[1]

Lines, polygons or circles are two-dimensional (abbreviated as "2-D" in the following) concepts, but more often people had to draw some real-world objects, which were mostly three-dimensional (abbreviated as "3-D" in the following). 3-D physical or abstract objects are even more interesting to consider, since diagrams had to be drawn on a flat surface, which is two-dimensional. Makers of those diagrams had to really choose which parts of the 3-D object must be put in the diagram, which to be left out, and what other extra effects to be added to the diagram so it might look like a 3-D object. For modern people, we are educated to represent 3-D objects on a flat face with perspective, and we are used to see those drawings in perspective as representations of 3-D objects. However, the development of perspective was rather late and mainly in Western Europe.[2] For earlier scholars who worked on solid geometry, such as Euclid or Archimedes, or for mathematics of non-Western cultures, such as East-Asian mathematics, scholars had ways to represent 3-D objects that are totally different from what we modern people are used to. Therefore, it would be very interesting for historians, mathematicians, and the public alike to see how ancient makers of diagrams represent 3-D physical or abstract objects. In this paper, the authors shall try to explore the topic of geometrical diagrams for 3-D objects in pre-modern cultures. A few examples shall be given about Greek geometry for the sake of discussion, but the main contents are about diagrams in Chinese, Korean, and Japanese mathematical texts, especially in the second millennium.

§2. Early geometrical diagrams, 2nd millennium BCE to 1st millennium CE

Geometrical diagrams were used in many different ways in ancient civilisations. Some of the earliest geometrical problems and diagrams could be found in ancient Mesopotamia. Figure 1 shows the Old Babylonian tablet BM 15285 (*ca.* 1750 BCE). There are many geometrical diagrams and problems attached to them. These diagrams also attest symmetrical designs in ancient art.[3]

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[1] Refer to, for example, Heath [4], p.153.
[2] For the history of perspective in Western Europe, see, for instance, Anderson [1].
[3] Robson [8], pp.93-113.

Figure 1. Old Babylonian tablet BM15285 collected in the British Museum.[4]

Greek mathematics inherited some knowledge from ancient Egyptian and Babylonian civilisations, but it is a well-known fact that Greek geometry was to a great extent influenced by Platonic philosophy and Aristotelian logic. Greek philosophers were suspicious about "moving" geometrical objects, as can be seen from the second and third propositions of Book I of Euclid's *Elements*. Euclid spent a great effort only to prove that one line segment could be moved to another place on the plane.[5] As a result, diagrams in Euclid's *Elements* are "static". Vertices of diagrams are labelled with letters, and usually different components of the diagrams are referred to with the labels,

4 https://mathoverflow.net/q/135201
5 See Heath [4], pp.244-247.

so their relations can be used to justify the propositions.

Compared to Greek mathematics, earlier Chinese diagrams (*tu* 圖), such as those related to the *Nine Chapters of Mathematical Art* (*Jiuzhang suanshu* [九章算術], 1st century CE, whose most important commentary were written by Liu Hui [劉徽] around 263 CE), seem to be used more often in a "dynamic" manner.[6] The most prominent case for manipulating diagrams dynamically are the two methods "out-in complimentary principle" (*Churu xiangbu* 出入相補) and "mending the void with the excess" (*yiying buxu* 以盈補虛).[7]

So far, what we have discussed here are all 2-D diagrams. In fact, many interesting 3-D diagrams in ancient Greek geometry survive to this day. Some of the most interesting pieces can be found in Archimedes' works. Figure 2 shows two kinds of cones that Archimedes drew in his works. As the reader could immediately tell, these diagrams do not resemble the cones a modern mathematician might draw on her thesis or a modern teacher on his blackboard, since we are trained to draw with perspective. On the left of Figure 2, the cone looks like an isosceles triangle with a height in it, but Θ is the vertex of the cone, ΘΛ is the height of the cone, and HK seem to be a diameter of the circle as the base of the cone. On the right of Figure 2, Δ is the vertex of the cone, while ΔA, ΔB, ΔΓ, ΔE, and ΔZ are line segments from the vertex to the circumference of the base.

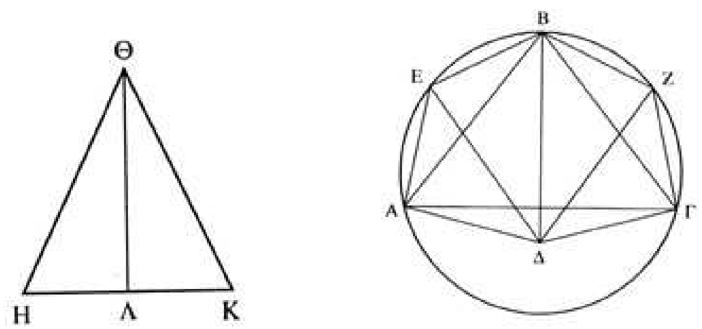

Figure 2. Two of Archimedes' diagrams for cones.[8]

[6] For a detailed analysis of the contents of the *Nine Chapters of Mathematical Arts* and its commentaries, refer to, for example, Chemla and Guo [3].

[7] For the Chinese methods of using diagrams dynamically, refer to, for example, Volkov [9] or to Guo [16], p. 185-200.

[8] The diagrams in Figure 2 were reconstructed by Reviel Netz from various versions ("codices") of Archimedes' famous work *On the Sphere and the Cylinder* (3rd century BCE). The reader may refer to Netz [6]. The two cones are shown in p.65 and p.106 in [6], respectively; for a general discussion of Archimedes' diagrams, refer to p.8 of [6].

Diagrams in Figure 2 show us that there could be different ways in representing 3-D objects in pre-modern times. Do we have 3-D diagrams from China in the first millennium? In fact, there is not any 1st-millennium Chinese geometrical diagram that survives to this day. The dynamic manner of using geometrical diagrams can only be interpreted from the passages and commentaries written in mathematical texts. All early diagrams in mathematical texts before 1000 CE in East Asian mathematical texts, if they ever existed, were lost.[9] Chinese geometrical diagrams surviving to this day came only from the second millennium. There are two main theories that explains this situation. The first is that when ancient Chinese were reasoning about geometrical propositions, they did not drew diagrams, but used flat material objects and solid blocks, and that is why early Chinese mathematical texts do not contain diagrams. This hypothesis is supported by Jean-Claude Martzloff and Karine Chemla.[10] The other theory is that mathematical diagrams were published separately from the main texts, and during the changes of the dynasties from the 8th to the 11th centuries, those books with geometrical diagrams were destroyed along with astronomical texts, because civilians were not allowed to collect astronomical texts and geometrical diagrams could easily be mistaken as astronomical charts. This hypothesis is supported by Alexei Volkov.[11]

Either way, the surviving geometrical diagrams in East Asia were all from the second millennium. We shall now discuss 2-D and 3-D Chinese diagrams in the second millennium, before we compare Korean and Japanese diagrams in the second half of the same millennium.

§3. Chinese diagrams for 2-D and 3-D objects in the 2nd millennium CE.

In the beginning of the second millennium CE, there are actually not so many existing Chinese mathematical texts that survive to this day, so our discussions of types of geometrical diagrams are still limited. Two of the most important mathematical treatises in the first half of the second millennium, the *Ceyuan haijing* [測圓海鏡] (Sea mirror of circle measurement, 1248) and the *Yigu yanduan* [益古演段] (Area pieces developments for the collection augmenting ancient knowledge, 1259), written by Li Ye [李冶] (1192–1279), contains only diagrams for two-dimensional objects.[12] In these

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[9] The Silla [新羅] kingdom on the Korean peninsula, and Japan as well, set up their state education and examination systems in the 8th century following those of the Tang Empire, and the mathematical textbooks used in Silla and Japan were not totally from China. Some texts might be their own creations, but those texts have been lost, too. For a quick comparison of the mathematics education systems and their textbooks for these three East Asian countries in the 8th century, refer to, for example, Jochi [13], p.76.

[10] Refer to Martzloff [5] and Chemla [2].

[11] Refer to Volkov [5].

[12] For the *Ceyuan haijing*, refer to Guo [14], vol.1, pp. 725-870. For the *Yigu yuanduan*, refer to Guo [14], vol. 1, pp. 871-942. For different interpretations and understandings for the “quadratic

two works, the diagrams are mainly used to set up problems of quadratic equations. Figure 3 are the two representative diagrams of the two texts.

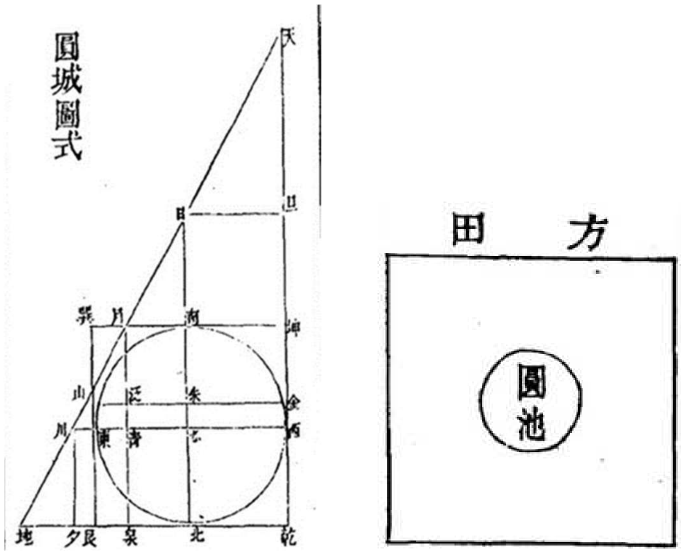

Figure 3. On the left, “Diagram for the Circular Castle” in the *Ceyuan haijing*.[13]
On the right, the diagram of a square filed with a circular pond in it, a representative diagram in the *Yigu yanduan*.[14]

The *Xiangjie jiuzhang suanfa* [詳解九章算法] (Detailed explanations for the calculation methods of the *Nine Chapters*, 1261) by Yang Hui [楊輝] (13th century) discusses many problems from the *Nine Chapters of Mathematical Art*, including problems of volumes for different solids, but no diagrams are attached to those problems. All geometrical diagrams are representations of 2-D objects.[15] In fact, the works of Yang Hui and Zhu Shijie [朱世傑] (mid-13th to early 14th centuries) in the 13th and 14th centuries also contain a rich variety of representation of two-dimensional objects such as different shapes of fields as well as root extraction processes, but they contain no diagram in sections of volume calculations.[16]

One mathematical treatise, also very important and contemporaneous to Li Ye’s and Yang Hui’s works, the *Shushu jiuzhang* [數書九章] (Mathematical treatise in nine chapters, 1247), does contain visual representations of 3-D objects, but these figures are usually “contextualised”, which means they are usually representations of real-world

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equations” in East Asian Mathematics, refer to, for example, Pollet and Ying [7].

[13] Refer to Guo [15], vol. 1, p. 732.

[14] Refer to Guo [15], vol. 1, p.883.

[15] For the *Xiangjie jiuzhang suanfa*, refer to Guo [15], vol.1, pp. 943-1044.

[16] For Zhu Shijie’s works, refer to Guo [15], vol. 1, pp.1119-1280.

situations instead of abstract geometrical diagrams.[17] Figure 4 shows two pictures of situations for problems of measurements.

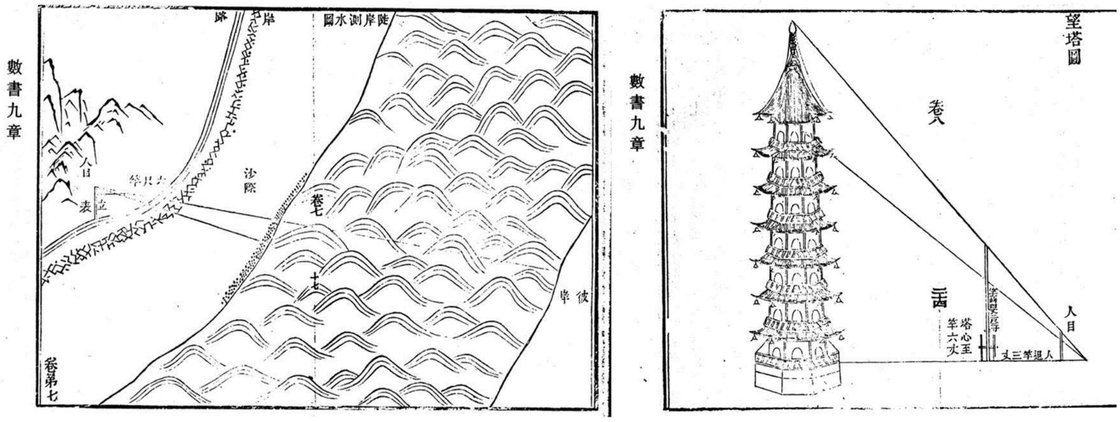
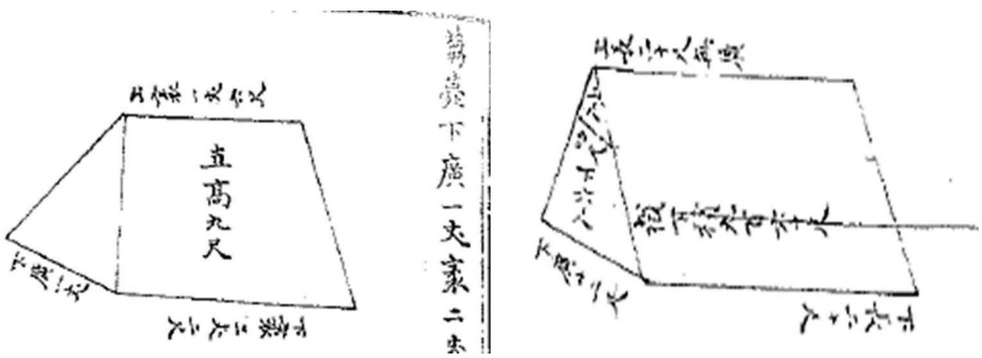


Figure 4. Contextualised representation of 3-D objects in situations of measurements in the *Shushu jiuzhang*.[18]

Although 3-D objects such as different types of pyramids and cones, and the sphere as well, had been discussed ever since the *Nine Chapters of Mathematical Art* compiled in the 1st century CE, no diagrams of those solids are attached to the texts discussing them before the 15th century CE, or none that survives to this day. Systematic representations of basic, and less "contextualised", solids appear as late as in the Ming Dynasty in the 16th century. The earliest diagrams about those "classical" solids that we can find are from the *Suanxue baojian* [算學寶鑑] (Precious mirror of mathematics, 1524) by Wang Wensu [王文素] (1465-?).[19] Beside the classical solids, Wang's work also gives pictures of other relatively realistic (but not exactly contextualised) solids. Figure 5 shows his diagrams for two classical solids from the *Nine Chapters of Mathematical Art*, and Figure 6 are two more relatively realistic solids.



[17] For the *Shushu jiuzhang*, refer to Guo [15], vol.1, pp.431-724.
[18] Refer to Guo [15], pp.521, 533.
[19] For the *Suanxue baojian*, refer to Guo [15], vol.2, pp.335-972.

Figure 5. Two classical solids in the *Suanxue baojian*.
Left: *chumeng* [芻甍]. Right: *qiandu* [塹堵].[20]

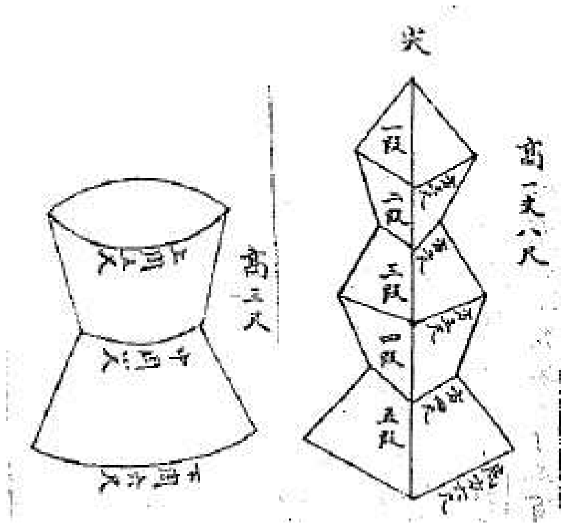


Figure 6. Drawings of relatively realistic objects in the *Suanxue baojian*.[21]

It is worth mentioning that although the entirety of the *Nine Chapters of Mathematical Art* and its more important commentaries were not available in China's Ming and early Qing Dynasties (15th to early 18th centuries), mathematicians such as Wang Wensu and Mei Wending [梅文鼎] seemed to have some knowledge of the contents of the ancient classic, which suggests that the mathematics of the ancient classic, if not most of the verbal and visual contents, managed to be preserved in the collective memory of pre-modern Chinese scholars. In fact, in the *Suanxue baojian*, it discusses almost all solids mentioned in chapter 5 of the *Nine Chapters of Mathematical Art*, and the volume formulas are essentially identical. Also, visual representations of the same objects are similar in the *Suanxue baojian* and in Mei's *Qiandu celiang* [塹堵測量] (Measurement with [the prism] *Qiandu*, written between 1701 and 1705).[22] Pre-modern Chinese diagrams for rectilinear 3-D objects essentially remain in the manner until the 19th century. In the next section, we shall discuss Korean and Japanese visual representations for 3-D objects in the same time period.

§4. Korean and Japanese diagrams for 3-D objects in the 17th to 19th centuries.

Although Korean and Japanese mathematics were initially influenced by their Chinese counterpart, both Chosŏn Korea and Edo Japan developed unique mathematical cultures and characteristics since the 17th century, called *tongsan* [東算] and *wasan* [和

[20] Refer to Guo [15], vol.2, pp.609, 904.
[21] Refer to Guo [15], vol.2, pp. 603-604.
[22] For the *Qiandu celiang*, refer to Guo [15], vol.4, pp. 655-682.

算], respectively.[23] There are relatively less Korean mathematical treatises survive to this day than Japanese ones, so our discussions about Korean visual representations would be more limited than Japanese ones. Nevertheless, we can still give some interesting examples for both cultures.

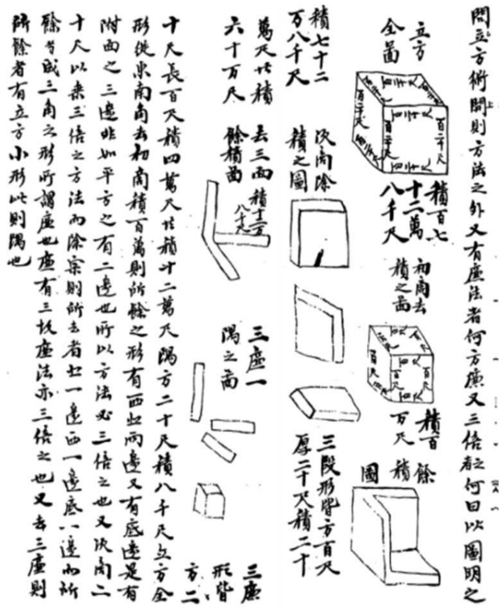


Figure 7. A set of diagrams explaining for cube root extractions in the *Chusŏ kwangyŏn*.[24]

Similar to Chinese treatises, most of the geometrical diagrams found in Korean mathematical treatises represent 2-D objects.[25] Only a few mathematical treatises contain any visual representations of 3-D objects. An immediate example is the set of typical diagrams explaining the extraction of cube roots in the *Chusŏ kwangyŏn* [籌書管見] (Humble view in a mathematical book, 1718) by Cho T'ae-Gu [趙泰耉] (1660-1723) in Figure 7.

The text that contain the richest collection of, and also the most interesting, representations of 3-D objects is the *Sŏgye swaerok* [書計瑣錄] (Fragmental Transcriptions of Writing and Calculations, 1786) by Pae Sang-Yŏl [裴相說] (1759-1789).[26] The first half of the text is about the linguistics and philology of Chinese and Korean languages, and only the second half is about mathematics. The paper [10] has thoroughly explored the diagrams for 3-D objects in Pae's work, but for the sake of discussions and comparisons, we still need to quote [10] and present some examples

[23] For a general overview of *tongsan*, refer to Kim-Kim [11]. For *wasan*, also refer to, for example, Jochi [13].
[24] Refer to Kim Yong-Woon [12], vol2, pp.151-152.
[25] For a general overview of geometrical diagrams in Korean mathematical texts, refer to Ying [10].
[26] A copy of the *Sŏgye swaerok* is published in Kim Yong-Woon [12], vol.4, pp.1-230.

below. On a first glance, many of the diagrams in Pae's work from the 18th century would not seem to represent any 3-D object in the eyes of the modern reader. To understand Pae's logic behind his drawings, we need to compare representations of both 2-D and 3-D objects.[27]

Figures 8.1, 8.2 and 8.3 are a series of rectilinear 2-D and 3-D objects. If we compare the two pairs of diagrams in Figures 8.1 and 8.2, we can see that the author put parallel line-segments on some (but not all) sides of 2-D diagrams (square, rectangle) to represent 3-D diagrams (cube, cuboid).

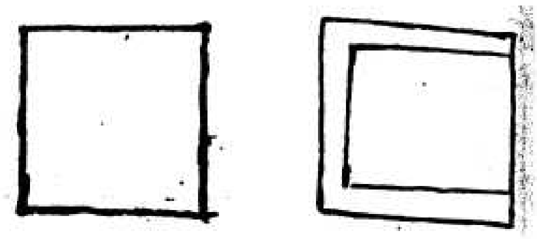


Figure 8.1. Left: square. Right: cube.

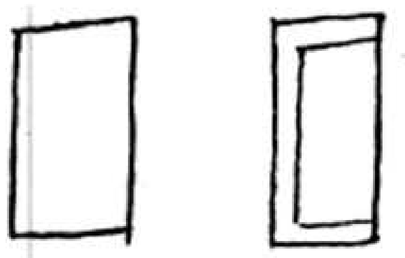


Figure 8.2. Left: rectangle. Right: cuboid.

The two diagrams in Figure 8.3 also have this characteristic. The author put line-segments on two sides of a triangle and a quadrilateral to represent a prism and a pyramid.

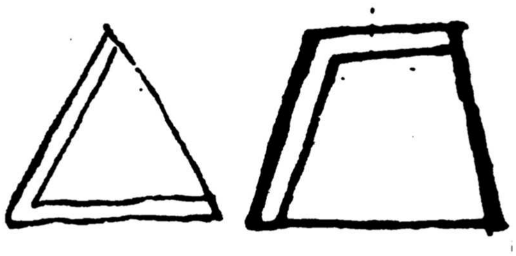


Figure 8.3. Left: triangular prism. Right: truncated pyramid.

[27] All diagrams are from Kim [12], vol.4, pp. 170-176. A thorough discussion about these diagrams can be found in Ying [10] and we rephrase a part of the discussion in this paper.

For curvilinear objects, a similar but somewhat different characteristic can also be seen. In Figure 8.4, parallel line-segments are added to all sides of 2-D shapes, so a rectangle can be transformed into a cylinder, and a triangle into a cone.

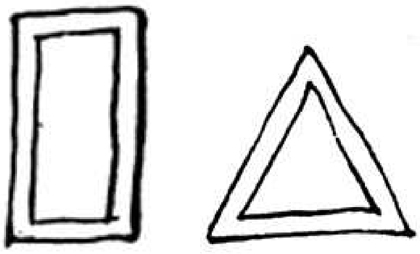


Figure 8.4. Left: cylinder. Right: cone.

Figure 8.5 is even more interesting. On the left of Figure 8.5, it is, to nobody's surprise, a circle, but on the right, it is a sphere! We are guessing that the author also wanted to use parallel lines to show it is not just a 2-D circle, but two concentric circles look as if it is a "ring", which is a common diagram in East Asian mathematical texts. So the author came up with an idea of puttng one big point in the centre to show it is a sphere.

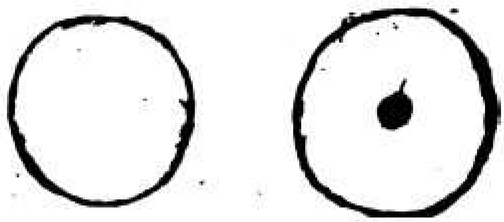


Figure 8.5. Left: circle. Right: sphere.

Pae Sang-Yŏl's creative ways to represent 3-D objects, such as the one for the sphere, are not unique examples. We can also see equally creative and interesting visual representations for 3-D objects in Japanese mathematics.

As in Chinese mathematics, earlier pictures of 3-D objects in *wasan* texts are often contextualised, such as those in *Jinkōki* [塵劫記] (Records from extremely small to extremely large numbers, 1634) and in *Kokon sanpōki* [古今算法記] (Records of ancient and contemporaneous mathematical methods, 1671). Figures 9.1 and 9.2 are two pages from each text that contain contextualised representations of 3-D objects.

Figure 9.1. Contextualised 3-D objects in the *Jinkōki*.[28]

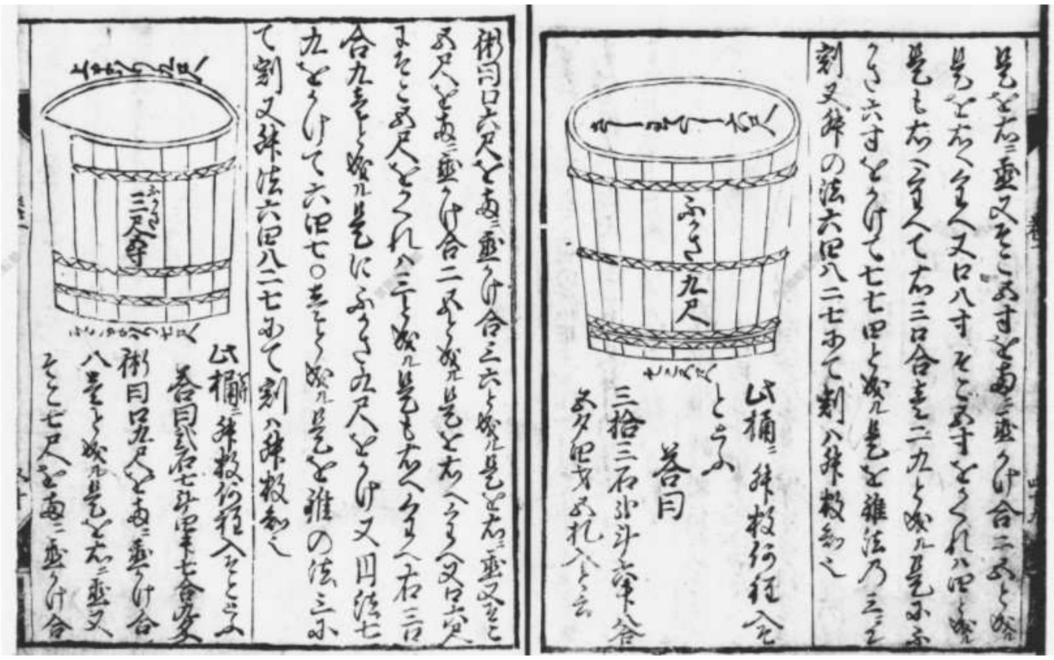


Figure 9.2. Contextualised 3-D objects in the *Kokon sanpōki*.[29]

[28] http://mahoroba.lib.nara-wu.ac.jp/y05/html/380/

[29] http://base1.nijl.ac.jp/iview/Frame.jsp?DB_ID=G0003917KTM&C_CODE=XSI6-005601

Generally speaking, rectilinear solids are depicted in *wasan* texts in a manner that is similar to how a modern teacher would draw on her blackboard. Figure 10 contains several examples of rectilinear diagrams from the *Enri sankei* [円理算経] (Mathematical Canon of *Enri*, 1842).

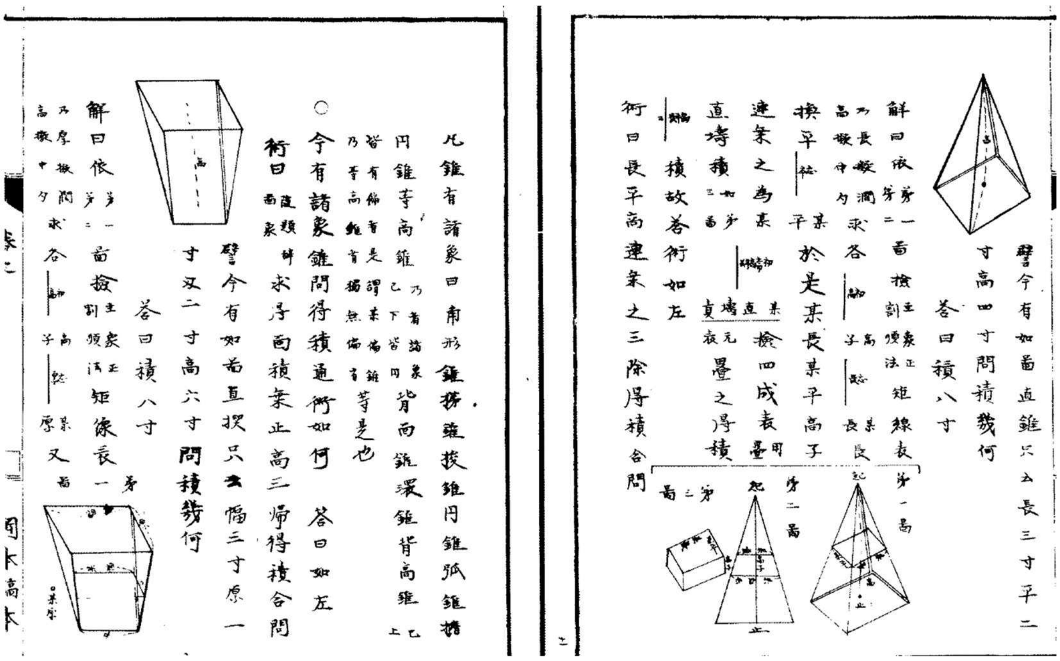


Figure 10. Rectilinear 3-D objects in the *Enri sankei*.[30]

However, curvilinear solids, such as the sphere, is more difficult to draw, and *wasan* mathematicians, as *tongsan* ones, had to find some creative ways to draw them. One method is using something similar to "shadows". Figure 11 is an example from the *Katsuyō sanpō* [括要算法] (Compendium of Mathematical Methods, 1709). In the Figure there is a ball (*litsugyoku* 立玉, literally "solid ball" or "solid orb"), on the left of which there is a piece of shadow to create a sense of the third dimension.[31]

30 東 北 大 學 和 算 資 料 デ ー タ ベ ー ス : http://www.i-repository.net/contents/tohoku/wasan/l/f004/21/f0042100131.png.

31 The authors want to thank the anonymous reviewer for reminding us to confirm the pronunciation of the term "立玉", since it is not used in modern times. We have checked several *wasan* texts of similar time periods but we could not find any *furigana* (reading aid) next to the kanji. Therefore, we transcribe it as "*litsugyoku*" according to the reviewer's suggestion.

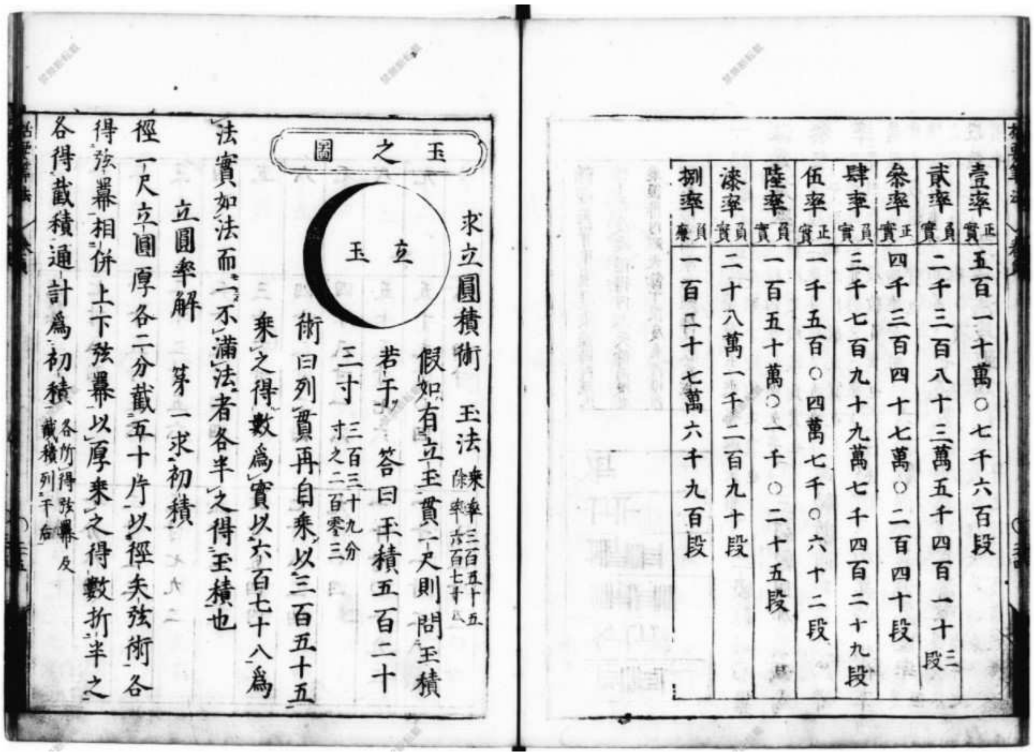


Figure 11. A sphere in the *Katsuyō sanpō*.[32]

Sometimes, sphere diagrams contain not "shadows" but only curves that try to create the sense of the third dimension, such as Figure 12 also from the *Enri sankei*. As the reader can see, the sphere (*litsuen* 立円, literally "solid circle") near the lower right corner is a simple diagram of a circle with a curve inside to show its third dimension, unlike the ball in Figure 11 that has a piece of shadow.

Also, if we take a look at the second and third column from the right in Figure 12, we can see that there are "eight faces" (*hachimen* 八面) on the second column and "eight solids" (*hattai* 八体) on the third. The only difference between the diagrams of the four curvilinear faces on the second column and those of the four curvilinear solids on the third is the curve in the middle of each diagram. It would seem that, at least for this author, it is a systematic way to represent 3-D objects.

[32] http://base1.nijl.ac.jp/iview/Frame.jsp?DB_ID=G0003917KTM&C_CODE=XSI6-001907.

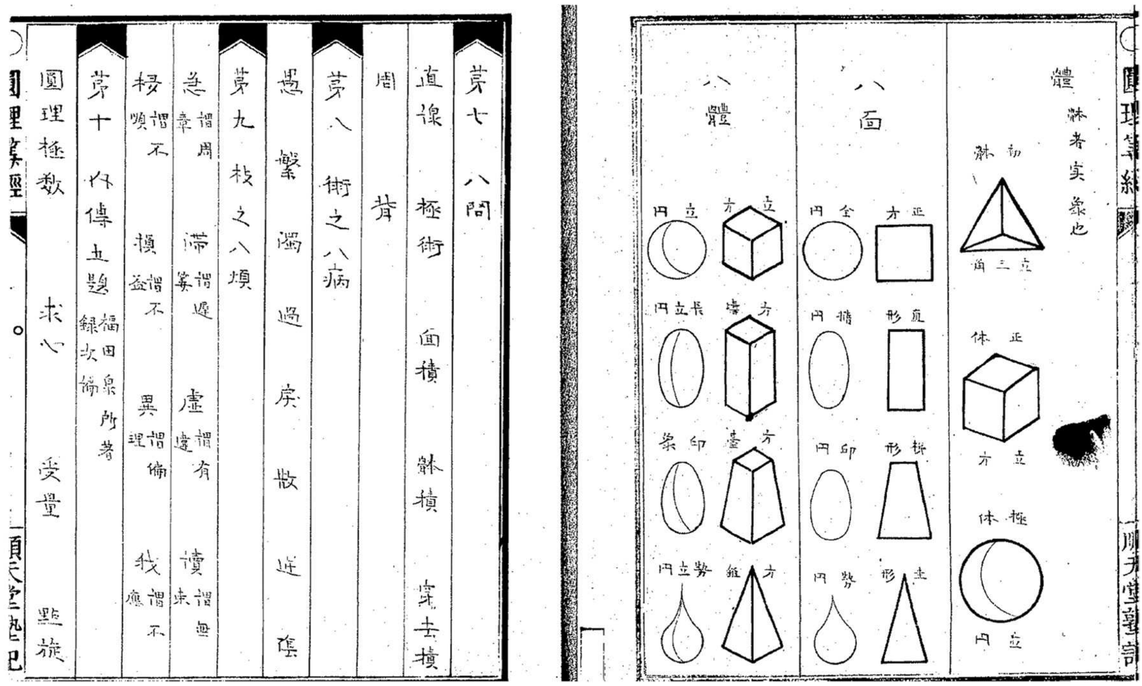


Figure 12. Diagrams for several 2-D and 3-D objects in the *Enri sankei*.[33]

As is well known to historians of mathematics and of Japanese culture, there is a group of cultural artefacts call "mathematical tablets" (*sangaku* 算額). They are wooden tablets mainly produced in the Edo period by mathematicians to be dedicated to Buddhist temples or Shinto shrines. Usually a *sangaku* contains one or several geometrical problems with coloured diagrams on it.[34] There are abundant instances of coloured representations of 3-D objects. Figure 13 shows the diagrams on a *sangaku* from Yamagata Prefecture dedicated in late Edo period. In the middle of the Figure, there is a diagram for a problem about a sphere inscribed with several smaller spheres and a hemisphere. As the reader can see, each sphere or hemisphere is divided by a curve into two regions with different colours. Moreover, it is actually difficult to see the darker side as shadow, because the darker sides are not all on the left or the right part of the sphere or the hemisphere. It would seem that the different colours on the same solid are meant to create a feeling of three dimensions, as if they were rectilinear solids with different "faces".

33 東北大學和算資料データベース：http://www.i-repository.net/contents/tohoku/wasan/l/f004/20/f0042000091.png?log=true&mid=4100002739&d=1511020186170

34 For a general discussion of *sangaku*, refer to, for example, Fukagawa and Rothman [14] (English original) and [14bis] (Japanese translation).

Figure 13. Diagrams on the *sangaku* in Mutsukunugi Hachiman Shrine of Yamagata City, Yamagata Prefecture dedicated in 1867.[35]

Thus, for curvilinear solids, both Korean and Japanese mathematicians used very creative ways to represent the third dimension on the flat surfaces of paper or wood plates

§5. Concluding remarks

So far in this paper we have discussed representations of 2-D and especially 3-D objects in the West and the East. Representing 3-D objects on a flat face has never been easy for human beings, and perspective drawing developed since the Renaissance is by no means the only way to do it, as we have seen from Archimedes' case. We have quoted prior studies to explain why there was no Chinese diagram in the first millennium or earlier that survives to this day. For Chinese diagrams in the second millennium, there are some difference for the early ones from the later. Earlier drawing of 3-D objects were usually contextualised, but later depictions became more abstract about pure geometrical solids. For *tongsan* (Korean) and *wasan* (Japanese) mathematicians in the 17^{th} to 19^{th} centuries, they used very creative ways to draw 3-D objects. In Korean mathematics, parallel line segments are used to show the third dimension, or the "depth". In Japanese mathematics, similar techniques are also used. *Wasan* mathematicians sometimes used shadows for the sphere. Equally often is a curve in the middle of a circle or an ellipse to show the third dimension, which is not unlike the Korean method of parallel line segments. For *sangaku* diagrams that allow colours, they put different colours on different regions divided by the middle curve, as if the two regions were two "faces", so the viewer could feel that the diagram represents a 3-D object. How much of these aforementioned creative practices of drawing 3-D objects was influenced by the introduction of European mathematics into East Asia remains to be investigated. And

[35] www.wasan.jp/yamagata/mutukunugi.html

we believe that there are still many interesting features of Chinese, Korean, and Japanese geometrical diagrams that we can study in the future.

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